

GENERALIZED THERMAL ZETA-FUNCTIONS

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February 1, 2008

Abstract

We calculate the partition function of a harmonic oscillator with quasi-periodic boundary conditions using the zeta-function method. This work generalizes a previous one by Gibbons and contains the usual bosonic and fermionic oscillators as particular cases. We give an alternative prescription for the analytic extension of the generalized Epstein function involved in the calculation of the generalized thermal zeta-functions. We also conjecture about the relation of our calculation to anyonic systems.

PACS: 03.65.-w; 03.65.Db; 05.30.-d.

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Since the generalized zeta-function method for computing determinants was introduced into theoretical physics calculations [1], this method has been applied successfully in different areas of physics. Examples vary from ordinary quantum and statistical mechanics to quantum field theory where anomalies, fermionic determinants as well as all sorts of effective actions can be computed by the generalized zeta-function prescription.

One of the first and very illustrative examples of this method, published in this Journal by Gibbons [2], concerns the computation of the partition functions of both the bosonic and fermionic oscillators. The application of the zeta-function method for these cases demands, the analytical extension of the so called inhomogeneous one-dimensional Epstein function (see for instance ref. [3] for some mathematical details on Epstein functions).

However, as far as we know, the interpolating system between the bosonic and fermionic oscillators has never been treated by the generalized zeta-function method in the literature. Recently, this problem has been solved by the authors using the Green function method [4].

In this letter, we generalize Gibbons work [2] since we compute $\det(\omega^2 - \partial_t^2)$ with a general boundary condition, which contains the periodic and antiperiodic boundary conditions as particular cases, by using the generalized zeta-function prescription. To do that, an analytical extension of what is called generalized inhomogeneous one-dimensional Epstein function is needed. We solve this problem generalizing the procedure of Ambjorn and Wolfran [5] to the usual Epstein function and obtaining this way a very convenient expression for the problem at hand.

Let us then calculate the partition function for a harmonic oscillator with quasi-periodic boundary conditions. For an usual (bosonic) harmonic oscillator and even for a fermionic oscillator the corresponding partition functions can be obtained by various methods, in particular by a path integral continued to the Euclidean space

$$\mathcal{Z}(\beta) = \int [d\mu_x] e^{-S} , \quad (1)$$

where S is the classical action. For the bosonic case we have $S = (1/2) \int_0^\beta x(\omega^2 - \partial_t^2)x dt$

and the integral is taken over periodic functions on the interval $(0, \beta)$:

$$x(t + \beta) = x(t) . \quad (2)$$

The fermionic case can be treated on similar grounds substituting the classical (c-number) functions x by anticommuting Grassmann variables x and \bar{x} , which must be antiperiodic on the interval $(0, \beta)$:

$$x(t + \beta) = -x(t) ; \quad \bar{x}(t + \beta) = -\bar{x}(t) \quad (3)$$

and the path integral measure is $[dx][d\bar{x}]$.

For the generalized case, however, there is no such a prescription. In order to circumvent this difficulty we follow the work of Gibbons who first applied the zeta-function method to the calculation of the partition functions for the bosonic and fermionic oscillators as the following determinants

$$\mathcal{Z}^{Bosonic}(\beta) = \det^{-1/2}(\omega^2 - \partial_t^2) \Big|_{periodic} \quad (4)$$

and

$$\mathcal{Z}^{Fermionic}(\beta) = \det^{+1}(\omega^2 - \partial_t^2) \Big|_{antiper.} . \quad (5)$$

As our starting point, we shall then take as the definition of the partition function for the general case, the following determinant

$$\mathcal{Z}_\sigma^\theta(\beta) = \det^\sigma(\omega^2 - \partial_t^2) \Big|_\theta = \det^\sigma(L)_\theta , \quad (6)$$

where σ is a parameter related to statistics and θ to the quasi-periodic boundary condition of the chosen oscillator. By quasi-periodic boundary condition we mean that

$$x(t + \beta) = e^{i\theta} x(t) . \quad (7)$$

The particular cases of the bosonic and fermionic oscillators can be obtained by taking $\sigma = -1/2$, $\theta = 0$ and $\sigma = +1$, $\theta = \pi$, respectively, as we shall show below.

The eigenvalues of the operator $L = \omega^2 - \partial_t^2$ under the quasi-periodic boundary condition are:

$$\lambda_m^\theta = \omega^2 + \left(\frac{\theta + 2m\pi}{\beta} \right)^2 ; \quad (m = 0, \pm 1, \pm 2, \dots) \quad (8)$$

and the corresponding generalized zeta-function is given by

$$\begin{aligned} \zeta^\theta(s; L) &\equiv \sum_m \left(\lambda_m^\theta \right)^{-s} \\ &= \sum_{m=-\infty}^{+\infty} \left[\omega^2 + \left(\frac{\theta + 2m\pi}{\beta} \right)^2 \right]^{-s} \\ &= \left(\frac{2\pi}{\beta} \right)^{-2s} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) , \end{aligned} \quad (9)$$

where $\nu = \omega\beta/2\pi$ and we defined

$$\mathcal{D}(s, \nu, \frac{\theta}{2\pi}) = \sum_{m=-\infty}^{+\infty} \left[\nu^2 + \left(\frac{\theta}{2\pi} + m \right)^2 \right]^{-s} . \quad (10)$$

The function $\zeta^\theta(s; L)$ is the generalized thermal zeta-function, which reduces to Gibbons cases for particular values of θ . From this function one can calculate the desired partition function

$$\mathcal{Z}_\sigma^\theta(\beta) = \exp \left\{ -\sigma \frac{\partial}{\partial s} \zeta^\theta(s; L) \right\} \Big|_{s=0} . \quad (11)$$

Once, the function defined by eq. (10) is not analytic at the origin we will have to make an analytic continuation for $\mathcal{D}(s, \nu, \frac{\theta}{2\pi})$ in order to determine the partition function $\mathcal{Z}_\sigma^\theta(\beta)$. In fact, the application of the generalized zeta-function method consists basically of the following three main steps: first find the eigenvalues and construct the series $\zeta(s) = \sum_m \lambda_m^{-s}$; then make an analytical extension for the whole complex s -plane (or at least into

a region containing the origin) and finally compute $\exp\left\{-\frac{\partial}{\partial s}\zeta(s)\right\}\Big|_{s=0}$. For the case at hand, the function $\mathcal{D}(s, \nu, \frac{\theta}{2\pi})$ is in fact related to the one-dimensional generalized Epstein function $Z_1^{c^2}(s; 1, a)$, since

$$Z_1^{c^2}(s; 1, a) = \sum_{n=1}^{+\infty} \left[c^2 + (n+a)^2 \right]^{-s}, \quad (12)$$

for which there are well known analytic continuations [6]. However, these formulas are not easy to handle and we performed a new representation for $\mathcal{D}(s, \nu, \frac{\theta}{2\pi})$ which can be considered as a generalization of the one given by Ambjorn and Wolfran [5]. In our case we obtain (see the appendix for details):

$$\mathcal{D}(s, \nu, \frac{\theta}{2\pi}) = \frac{\sqrt{\pi}}{\Gamma(s)} \left[\frac{\Gamma(s - \frac{1}{2})}{\nu^{2s-1}} + 4 \sum_{m=1}^{+\infty} \cos(m\theta) \left(\frac{m\pi}{\nu} \right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2m\pi\nu) \right], \quad (13)$$

where $K_\alpha(x)$ is the modified Bessel function of the second kind with order α . From expression (13) it can also be shown that (see the appendix)

$$\mathcal{D}(0, \nu, \frac{\theta}{2\pi}) = 0 \quad (14)$$

and

$$\frac{\partial}{\partial s} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) \Big|_{s=0} = -\ln [2(\cosh 2\pi\nu - \cos \theta)]. \quad (15)$$

Substituting (15) into (11), we have:

$$\begin{aligned} \mathcal{Z}_\sigma^\theta(\beta) &= \exp \left[-\sigma \frac{\partial}{\partial s} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) \right] \Big|_{s=0} \\ &= \exp \{ \sigma \ln [2(\cosh 2\pi\nu - \cos \theta)] \} \\ &= 4^\sigma \left[\cosh^2 \frac{\omega\beta}{2} - \cos^2 \frac{\theta}{2} \right]^\sigma. \end{aligned} \quad (16)$$

This partition function will reduce naturally to the corresponding one of the usual bosonic oscillator if we take $\sigma = -1/2$ and $\theta = 0$ [2],

$$\mathcal{Z}^{Bosonic}(\beta) = \left(2 \sinh \frac{\omega\beta}{2} \right)^{-1}. \quad (17)$$

Analogously, for the quadratic fermionic oscillator of Finkelstein and Villasante [7] we have, after choosing $\sigma = +1, \theta = \pi$, that

$$\mathcal{Z}_{Quadratic}^{Fermionic}(\beta) = 4 \cosh^2 \frac{\omega\beta}{2}. \quad (18)$$

The linear fermionic oscillator [8] is obtained with the choice $L \rightarrow L^{1/2}; \sigma = +1, \theta = \pi$

$$\mathcal{Z}_{Linear}^{Fermionic}(\beta) = 2 \cosh \frac{\omega\beta}{2}. \quad (19)$$

In fact, because of the general properties of determinants, the above choice could also be $\sigma = +1/2, \theta = \pi$ and L unchanged. Curiously, Gibbons was able to find this result for the linear fermionic oscillator without a similar rescaling in L . This was possible since he considered, from the beginning, the sum in the zeta-function (9) only over positive values of m . This naturally falls into our result since this introduces a factor 1/2 for $\mathcal{D}(s, \nu, \frac{\theta}{2\pi})$ and results in a corresponding power for the determinant, as can be seen, for example, in eq. (15).

A physical interpretation of our result is also possible. Equation (16) corresponds to the partition function of what we call the anyonic oscillator, in the sense that, besides interpolating continuously between the bosonic and fermionic cases it contains these two cases as particular choices of the parameters involved. This result coincides with the one given by the authors using the Green function method to calculate the ratio of two determinants [4].

Actually, anyons are expected to live in two space dimensions [9], but the generalized statistics they are related to, can also be defined in one space dimension [10] as we have done here. Furthermore, it has been shown in the literature that anyons confined to the lowest Landau level correspond to anyons in one dimension [11].

As a satisfactory quantization for anyonic systems are not known yet, we believe that our result may contribute towards this direction. Besides, we hope that our analytic con-

tinuation for $\mathcal{D}(s, \nu, \frac{\theta}{2\pi})$ will be of help in discussing other problems, specially in quantum field theory as the computation of effective actions at finite temperature, the Casimir effect, etc.

Acknowledgements. The authors were partially supported by CNPq, Brazilian agency.

Appendix

Here, we are going to derive the analytic continuation for $\mathcal{D}(s, \nu, \frac{\theta}{2\pi})$ given by eq. (13), in the lines of Ambjorn and Wolfran [5] and derive its main properties at $s = 0$. In fact their result corresponds to a particular case of ours, when we take $\theta = 0$.

Starting from eq. (10) and using the integral representation for the Gamma function [12]

$$\Gamma(s)A^{-s} = \int_0^\infty d\tau \tau^{s-1} e^{-A\tau}, \quad (20)$$

which is valid for $\text{Re } \tau > 0$ we find

$$\begin{aligned} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) &= \sum_{m=-\infty}^{+\infty} \left[\nu^2 + \left(\frac{\theta}{2\pi} + m \right)^2 \right]^{-s} \\ &= \frac{1}{\Gamma(s)} \sum_{m=-\infty}^{+\infty} \int_0^\infty dx x^{s-1} \exp \left\{ - \left[\nu^2 + \left(\frac{\theta}{2\pi} + m \right)^2 \right] x \right\} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty dx x^{s-1} \exp \left\{ -\nu^2 x \right\} \sum_{m=-\infty}^{+\infty} \exp \left\{ - \left(\frac{\theta}{2\pi} + m \right)^2 x \right\} \quad (21) \end{aligned}$$

Now, using the Poisson summation formula [13]

$$\sum_{n=-\infty}^{+\infty} e^{-(z+n)^2 \frac{\pi}{\tau}} = \sqrt{\tau} \sum_{n=-\infty}^{+\infty} e^{-n^2 \pi \tau - 2inz\pi} \quad (22)$$

and identifying $m = n$, $z = \theta/2\pi$, $x = \pi/\tau$ we can write

$$\begin{aligned}
\mathcal{D}(s, \nu, \frac{\theta}{2\pi}) &= \frac{1}{\Gamma(s)} \int_0^\infty dx x^{s-1} \exp\left\{-\nu^2 x\right\} \sqrt{\frac{\pi}{x}} \sum_{m=-\infty}^{+\infty} \exp\left\{-m^2 \frac{\pi^2}{x} - im\theta\right\} \\
&= \frac{\sqrt{\pi}}{\Gamma(s)} \sum_{m=-\infty}^{+\infty} e^{-im\theta} \int_0^\infty dx x^{s-\frac{1}{2}-1} e^{-\nu^2 x - m^2 \frac{\pi^2}{x}}.
\end{aligned} \tag{23}$$

The integral in the last line of the above equation can be written in terms of a modified Bessel function of the second kind, $K_\alpha(y)$ [12]

$$\int_0^\infty dx x^{\alpha-1} e^{-\gamma x - \frac{\beta}{x}} = 2 \left(\frac{\beta}{\gamma}\right)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{\beta\gamma}), \tag{24}$$

which is valid for $\text{Re}\beta > 0$ and $\text{Re}\gamma > 0$. Then, splitting the sum over m from $-\infty$ to $+\infty$ into the sums over negative values, $m = 0$ and positive values and identifying $\alpha = s - 1/2$, $\beta = m^2\pi^2$ and $\gamma = \nu^2$, we have

$$\mathcal{D}(s, \nu, \frac{\theta}{2\pi}) = \frac{G(s)}{\Gamma(s)}, \tag{25}$$

where we defined

$$G(s) = \sqrt{\pi} \left\{ \frac{\Gamma(s - \frac{1}{2})}{\nu^{2s-1}} + 4 \sum_{m=1}^{+\infty} \cos(m\theta) \left(\frac{m\pi}{\nu}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2m\pi\nu) \right\}, \tag{26}$$

which is the result expressed in eq. (13).

Now, we are going to compute $\mathcal{D}(0, \nu, \frac{\theta}{2\pi})$ as well as $\partial\mathcal{D}(0, \nu, \frac{\theta}{2\pi})/\partial s$. Noting that $G(s)$ is an entirely analytical function and that $\Gamma(s)$ diverges as $s \rightarrow 0$, we see that

$$\mathcal{D}(0, \nu, \frac{\theta}{2\pi}) = 0. \tag{27}$$

Next, we turn to the derivative of the zeta-function $\zeta^\theta(s; L)$ at the origin,

$$\begin{aligned}
\frac{\partial}{\partial s} \zeta^\theta(s; L) \Big|_{s=0} &= \frac{\partial}{\partial s} \left\{ \left(\frac{2\pi}{\beta}\right)^{-2s} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) \right\} \Big|_{s=0} \\
&= \frac{\partial}{\partial s} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) \Big|_{s=0},
\end{aligned} \tag{28}$$

where we already used the fact that $\mathcal{D}(0, \nu, \frac{\theta}{2\pi}) = 0$. Using eq.(25) we have

$$\frac{\partial}{\partial s} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) \Big|_{s=0} = - \left[\frac{\Gamma'(s)}{\Gamma^2(s)} G(s) \right] \Big|_{s=0} + \left[\frac{1}{\Gamma(s)} G'(s) \right] \Big|_{s=0} \quad (29)$$

and the second term of the r.h.s. of the above expression vanishes too. Noting that

$$\lim_{s \rightarrow 0} \frac{\Gamma'(s)}{\Gamma^2(s)} = -1$$

and using the fact that $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, we have

$$\begin{aligned} \frac{\partial}{\partial s} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) \Big|_{s=0} &= G(0) \\ &= -2\pi\nu + 4 \sum_{m=1}^{+\infty} \cos(m\theta) \left(\frac{\nu}{m}\right)^{\frac{1}{2}} K_{-\frac{1}{2}}(2m\pi\nu) . \end{aligned} \quad (30)$$

Then, using the well known expression for $K_{-\frac{1}{2}}(z)$, namely, [12]

$$K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \quad (31)$$

and the result

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{-2\pi\nu n} = \pi\nu - \ln[2 \sinh(\pi\nu)] , \quad (32)$$

we finally obtain

$$\begin{aligned} \frac{\partial}{\partial s} \mathcal{D}(s, \nu, \frac{\theta}{2\pi}) \Big|_{s=0} &= -2\pi\nu + 2 \sum_{m=1}^{+\infty} \cos(m\theta) \frac{1}{m} e^{-2m\pi\nu} \\ &= -2\pi\nu + \sum_{m=1}^{+\infty} \frac{1}{m} e^{-(2\pi\nu-i\theta)m} + \sum_{m=1}^{+\infty} \frac{1}{m} e^{-(2\pi\nu+i\theta)m} \\ &= -\ln \left[2 \sinh \left(\pi\nu + \frac{i\theta}{2} \right) \right] \\ &= -\ln [2 (\cosh(2\pi\nu) - \cos\theta)] , \end{aligned} \quad (33)$$

which is precisely the result shown in eq.(15).

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